Liouville conformal blocks and representations of the mapping class group

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Liouville conformal field theory

Let Σ be a Riemann surface. We are interested in the path integral

$$\langle F \rangle = \int_{\phi:\Sigma\to\mathbb{R}} F(\phi)\, e^{-S(\phi)}\, \mathrm{D}\phi$$

where S is the Liouville action

$$S(\phi) = \frac{1}{4\pi} \int_\Sigma \left(|d\phi|^2 + QK\phi + 4\pi\mu e^{\gamma\phi} \right) |\mathrm{d}z|^2$$

and F is a product of vertex operators

$$F(\phi) = \prod_{j=1}^{N} e^{\alpha_j \phi(z_j)}$$

Here $0 < \gamma < 2$, $Q = \frac{\gamma}{2} + \frac{2}{\gamma}$, $\mu > 0$, $\alpha_j \in \mathbb{R}$, $z_j \in \Sigma$.

Conformal bootstrap

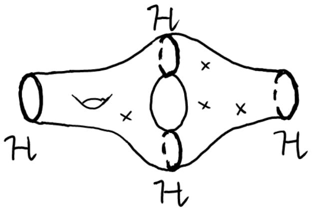
Idea: Decompose the surface.

$$\int_{\varphi:\mathcal{C}\to\mathbb{R}} \left(\int_{\substack{\phi_1:\Sigma_1\to\mathbb{R} \\ \phi_1|_{\mathcal{C}}=\varphi}} F_1(\phi_1)\, e^{-S_{\Sigma_1}(\phi_1)}\, \mathrm{D}\phi_1 \right) \left(\int_{\substack{\phi_2:\Sigma_2\to\mathbb{R} \\ \phi_2|_{\mathcal{C}}=\varphi}} F_2(\phi_2)\, e^{-S_{\Sigma_2}(\phi_2)}\, \mathrm{D}\phi_2 \right) \mathrm{D}\varphi$$

$$= \int_{\phi:\Sigma\to\mathbb{R}} F_1(\phi|_{\Sigma_1})\, F_2(\phi|_{\Sigma_2})\, e^{-S_\Sigma(\phi)}\, \mathrm{D}\phi$$

Segal's axioms

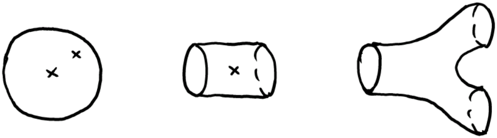
To each circle $\mathcal{C}$, associate a Hilbert space $\mathcal{H} = L^2(\{\varphi : \mathcal{C} \to \mathbb{R}\})$.
To each surface Σ, associate an amplitude operator $\mathcal{A}_\Sigma : \mathcal{H}^{b_-} \to \mathcal{H}^{b_+}$.



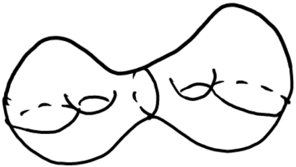
Gluing corresponds to composing the amplitudes.

In mathematical language: This defines a projective functor from the cobordism category of surfaces to the category of Hilbert spaces.

Conformal blocks



Any surface can be decomposed into these primitives.



Conformal blocks

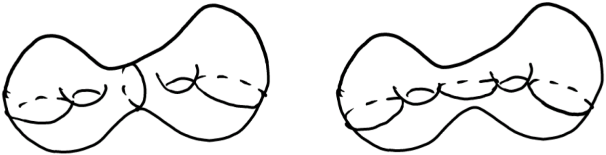
One can show that, for a general surface Σ, there exists a function $\mathcal{F}_\Sigma : \mathbb{R}_+^M \to \mathbb{C}$ such that

$$\langle F \rangle = C_0 \int_{\mathbb{R}_+^M} |\mathcal{F}_\Sigma(\mathbf{p})|^2 \, \rho(\mathbf{p}) \, \mathrm{d}\mathbf{p}$$

where M is the number of cuts, C_0, ρ are explicit and do not depend on Σ.

This function $\mathcal{F}_\Sigma$ depends on how you cut the surface!

Dependence on cutting



This defines a projective representation of the Moore–Seiberg groupoid (and in particular the mapping class group) on the Hilbert space $L^2(\mathbb{R}_+^M)$.

Connection with quantum Teichmüller theory

Conjecture (Teschner)

The space $L^2(\mathbb{R}_+^M)$ should be thought of as a quantum Teichmüller space, and the representation from Liouville CFT should be equivalent to another representation coming from the theory of the quantization of Teichmüller space (due to Fock, Chekhov, Kashaev from the end of the 90s).

State of the art:

- The rigorous definition of Liouville CFT is done in the last decade (due to David, Guillarmou, Rhodes, Kupiainen, Vargas).
- The definition of the representation from Liouville CFT is work in progress.
- The rigorous definition of the tool used in the presumed proof (quantized length operators) is work in progress.

Thank you!