Introduction to Compactified Imaginary Liouville Theory

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Let (Σ, g) be a Riemannian surface. The N-point correlation function of **Liouville Conformal Field Theory** is defined by the path integral

$$\left\langle \prod_{j=1}^{N} V_{\alpha_j}(z_j) \right\rangle_{\Sigma,g} = \int_{\phi:\Sigma\to\mathbb{R}} \prod_{j=1}^{N} V_{\alpha_j}(z_j)\, e^{-S(\phi)}\, \mathrm{D}\phi$$

where S is the Liouville action

$$S(\phi) = \frac{1}{4\pi} \int_{\Sigma} \left(|d\phi|_g^2 + QK_g\phi + 4\pi\mu e^{\gamma\phi} \right) \mathrm{d}v_g$$

and the $V_{\alpha_j}(z_j)$ $(z_j \in \Sigma)$ are the vertex operators

$$V_{\alpha_j}(z_j) = e^{\alpha_j \phi(z_j)}$$

Here $\mu > 0$, $0 < \gamma < 2$, $Q = \frac{\gamma}{2} + \frac{2}{\gamma}$, $\alpha_j \in \mathbb{R}$.

Imaginary Liouville Theory, Compactified

Let (Σ, g) be a Riemannian surface. The N-point correlation function of **Compactified Imaginary Liouville Theory** is defined by the path integral

$$\left\langle \prod_{j=1}^{N} V_{\alpha_j}(z_j) \right\rangle_{\Sigma,g} = \int_{\phi:\Sigma\to\mathbb{R}/2\pi R\mathbb{Z}} \prod_{j=1}^{N} V_{\alpha_j}(z_j)\, e^{-S(\phi)}\, \mathrm{D}\phi$$

where S is the Liouville action

$$S(\phi) = \frac{1}{4\pi}\int_\Sigma \left(|d\phi|_g^2 + \mathrm{i}QK_g\phi + 4\pi\mu e^{\mathrm{i}\beta\phi}\right) \mathrm{d}v_g$$

and the $V_{\alpha_j}(z_j)$ $(z_j \in \Sigma)$ are the vertex operators

$$V_{\alpha_j}(z_j) = e^{\mathrm{i}\alpha_j\phi(z_j)}$$

Here $\mu \in \mathbb{C}$, $0 < \beta < \sqrt{2}$, $Q = \frac{\beta}{2} - \frac{2}{\beta}$, $\alpha_j \in \mathbb{R}$, $R > 0$.

Consequences of the Compactification

We decompose the field $\phi = c + X_g$ where X_g is the part with mean zero. With respect to the zero mode c, the path integral is of the form

$$\int e^{\mathrm{i}Ac-\mu Be^{\mathrm{i}\beta c}} \mathrm{d}c$$

where

$$A = \sum \alpha_j - Q\chi(\Sigma)$$
$$B = \int_\Sigma e^{\mathrm{i}\beta X_g} \, \mathrm{d}v_g$$

This integral exists and is well-defined if integrated over $c \in \mathbb{R}/2\pi R\mathbb{Z}$ where the *compactification radius* $R > 0$ is chosen so that

$$\beta R, QR, \alpha_j R \in \mathbb{Z}$$

Consequences on the CFT structure:

- This CFT is *non-unitary*.
- The conditions $\beta R, QR \in \mathbb{Z}$ together imply that the central charge $c = 1 - 6Q^2$ is *rational*.
- The condition $\alpha_j R \in \mathbb{Z}$ implies that the spectrum is *discrete*.
- The correlation function is trivial unless the *neutrality condition*

$$\sum \alpha_j - Q\chi(\Sigma) + p\beta = 0$$

  is satisfied for some $p \in \mathbb{N}_{\geq 0}$, in which case it has the form

$$2\pi R \frac{(-\mu)^p}{p!} \mathbb{E}\left[\left(\int_\Sigma e^{\mathrm{i}\beta X_g} \, \mathrm{d}v_g \right)^p \right]$$

Construction 1: Topological Ingredient

A map $\Sigma \to \mathbb{R}/2\pi R\mathbb{Z}$ is not necessarily contractible. Consider

$$[\Sigma, \mathbb{R}/2\pi R\mathbb{Z}] = \left\{\text{free homotopy classes } \Sigma \to \mathbb{R}/2\pi R\mathbb{Z}\right\}$$

We have an isomorphism of abelian groups

$$\begin{aligned} [\Sigma, \mathbb{R}/2\pi R\mathbb{Z}] &\cong H^1(\Sigma; \mathbb{Z}) \\ [\phi] &\mapsto [\tfrac{1}{2\pi R} d\phi] \\ [I_{x_0}(2\pi R\omega)] &\leftarrow\!\shortmid [\omega] \end{aligned}$$

where we fix a basepoint $x_0 \in \Sigma$ and define

$$\begin{aligned} I_{x_0}(2\pi R\omega) : \Sigma &\to \mathbb{R}/2\pi R\mathbb{Z} \\ x &\mapsto \int_{x_0}^{x} 2\pi R\omega \end{aligned}$$

The Compactified Free Field

We integrate over

$$\phi_g = c + X_g + I_{x_0}(2\pi R\omega)$$

where

- $c \in \mathbb{R}/2\pi R\mathbb{Z}$,
- X_g is the Gaussian Free Field with mean zero,
- $[\omega] \in H^1(\Sigma;\mathbb{Z})$.

Changing the basepoint x_0 or the representative ω in its cohomology class amounts to translations in c and X_g.

One can view this as a random field in the space $\mathcal{D}'(\Sigma, \mathbb{R}/2\pi R\mathbb{Z})$ of $(\mathbb{R}/2\pi R\mathbb{Z})$-valued generalized functions on Σ.

We define

$$\langle F\rangle^0_{\Sigma,g} := \int_{\phi:\Sigma\to\mathbb{R}/2\pi R\mathbb{Z}} F(\phi)\, e^{-\frac{1}{4\pi}\int_\Sigma |d\phi|_g^2 \mathrm{d}v_g}\, \mathrm{D}\phi := \sqrt{\frac{\mathrm{vol}_g \Sigma}{\det' \Delta_g}} \times$$

$$\sum_{[\omega]\in H^1(\Sigma;\mathbb{Z})} \int_0^{2\pi R} \mathbb{E}\left[F(\phi_g)\, e^{-\pi R^2 \int_\Sigma |\omega|_g^2 \,\mathrm{d}v_g - R\int_\Sigma X_g\, d^*\omega \,\mathrm{d}v_g}\right] \mathrm{d}c$$

where $\det' \Delta_g$ is the regularized determinant of the Laplacian Δ_g and the expectation is over the GFF X_g.

Proposition

This is well-defined (if it converges). For $\rho \in C^\infty(\Sigma, \mathbb{R})$,

$$\langle F\rangle^0_{\Sigma,e^\rho g} = \langle F\rangle^0_{\Sigma,g}\, e^{\frac{1}{96\pi}\int_\Sigma \left(|d\rho|_g^2 + 2K_g\rho\right) \mathrm{d}v_g}$$

Construction 2: Curvature Term

We need to make sense of the integral

$$\int_{\Sigma} K_g(c + X_g + I_{x_0}(2\pi R\omega))\,\mathrm{d}v_g$$

The problematic term is $I_{x_0}(2\pi R\omega)$ which is multi-valued as a map to $\mathbb{R}$.

Idea: Integrate over a domain of full measure with trivial H_1 (so that $I_{x_0}(2\pi R\omega)$ is well-defined).

Let $\sigma = (a_1, b_1, \ldots, a_{\mathfrak{g}}, b_{\mathfrak{g}})$ be a family of simple closed curves (disjoint from x_0) representing a symplectic basis of $H_1(\Sigma)$. We define

$$\int_\Sigma^\sigma K_g I_{x_0}(2\pi R\omega)\,\mathrm{d}v_g := \int_{\Sigma\setminus\cup\sigma} K_g I_{x_0}(2\pi R\omega)\,\mathrm{d}v_g + 4\pi R \sum_{j=1}^{\mathfrak{g}} \left(\int_{a_j} \omega \int_{b_j} k_g\,\mathrm{d}\ell_g - \int_{b_j} \omega \int_{a_j} k_g\,\mathrm{d}\ell_g \right)$$

where k_g is the geodesic curvature, ℓ_g is the length, $\mathfrak{g}$ is the genus of Σ.

Proposition

For two such families σ and σ',

$$\int_\Sigma^{\sigma'} K_g I_{x_0}(2\pi R\omega)\,\mathrm{d}v_g - \int_\Sigma^{\sigma} K_g I_{x_0}(2\pi R\omega)\,\mathrm{d}v_g \in 8\pi^2 R\mathbb{Z}$$

In particular, $e^{-\frac{1}{4\pi}Q\int_\Sigma^\sigma K_g I_{x_0}(2\pi R\omega)\,\mathrm{d}v_g}$ is well-defined.

Construction 3: Exponential Terms

The Liouville potential

$$\int_\Sigma e^{\mathrm{i}\beta(c+X_g+I_{x_0}(2\pi R\omega))}\,\mathrm{d}v_g$$

can be defined by the *imaginary* Gaussian Multiplicative Chaos

$$M_g^\beta(X_g,\mathrm{d}v_g) := \lim_{\varepsilon\to 0} \varepsilon^{-\beta^2/2} e^{\mathrm{i}\beta X_{g,\varepsilon}}$$

where $X_{g,\varepsilon}$ is the ε-regularized GFF.

The vertex operators $e^{\mathrm{i}\alpha_j\phi(z_j)}$ are defined similarly. We assume $\alpha_j > Q$ for convergence.

Weyl Anomaly

Theorem

This theory is *conformal* in the following sense. For $\rho \in C^\infty(\Sigma, \mathbb{R})$,

$$\left\langle \prod V_{\alpha_j}(z_j) \right\rangle_{\Sigma, e^\rho g} = \left\langle \prod V_{\alpha_j}(z_j) \right\rangle_{\Sigma, g} e^{\frac{c}{96\pi} \int_\Sigma \left(|d\rho|_g^2 + 2K_g \rho \right) \mathrm{d}v_g - \sum \Delta_{\alpha_j} \rho(z_j)}$$

where

$$c = 1 - 6Q^2$$

is the *central charge* of the theory and

$$\Delta_{\alpha_j} = \frac{\alpha_j}{2} \left(\frac{\alpha_j}{2} - Q \right)$$

is the *conformal dimension* of $V_{\alpha_j}(z_j)$.

Generalization 1: Spin

We can integrate over fields $\phi : \Sigma \to \mathbb{R}/2\pi R\mathbb{Z}$ with prescribed winding number $m_j \in \mathbb{Z}$ around $z_j \in \Sigma$. Then we additionally specify a nonzero tangent vector v_j at z_j to make sense of

$$e^{\alpha_j \phi(z_j)} := \lim_{t \to 0^-} e^{\alpha_j \phi(z_j + t v_j)}$$

This defines an *electromagnetic operator* $V_{\alpha_j, m_j}(v_j)$ with *electric charge* α_j and *magnetic charge* m_j. Its conformal dimension is

$$\Delta_{\alpha_j, m_j} = \frac{\alpha_j}{2}\left(\frac{\alpha_j}{2} - Q\right) + \frac{m_j^2}{4} R^2$$

Proposition

For $(\theta_1, \ldots, \theta_N) \in \mathbb{R}^N$,

$$\left\langle \prod V_{\alpha_j, m_j}(e^{\mathrm{i}\theta_j} v_j) \right\rangle_{\Sigma, g} = \left\langle \prod V_{\alpha_j, m_j}(v_j) \right\rangle_{\Sigma, g} e^{\mathrm{i}R \sum (\alpha_j - Q) m_j \theta_j}$$

Generalization 2: Boundary Version

For a Riemann surface Σ with boundary, we can integrate over fields $\phi : \Sigma \to \mathbb{R}/2\pi R\mathbb{Z}$ satisfying the Neumann boundary condition

$$\frac{\partial}{\partial \nu}\Big|_{\partial\Sigma} \phi = 0$$

where ν is a normal vector along $\partial\Sigma$. This defines a *Boundary CFT*.

The functional in this case is

$$S(\phi) = \frac{1}{4\pi}\int_\Sigma |d\phi|_g^2 \, \mathrm{d}v_g + \frac{\mathrm{i}Q}{4\pi}\int_\Sigma K_g \phi \, \mathrm{d}v_g + \frac{\mathrm{i}Q}{2\pi}\int_{\partial\Sigma} k_g \phi \, \mathrm{d}\ell_g + \mu \int_\Sigma e^{\mathrm{i}\beta\phi} \, \mathrm{d}v_g + \int_{\partial\Sigma} \mu_\partial \, e^{\mathrm{i}\frac{\beta}{2}\phi} \, \mathrm{d}\ell_g$$

Generalization 3: Amplitudes à la Segal

For a Riemann surface Σ with boundary, we can integrate over fields $\phi : \Sigma \to \mathbb{R}/2\pi R\mathbb{Z}$ with prescribed boundary values

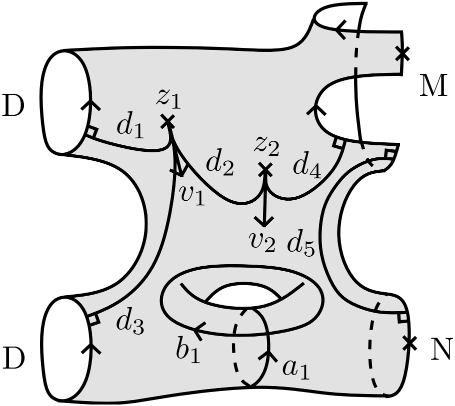
$$\phi\big|_{\partial_j \Sigma} = \varphi_j$$

The resulting quantity is the integral kernel of an *amplitude operator* between powers of the Hilbert space $\mathcal{H} = L^2(\{\mathbb{S}^1 \to \mathbb{R}/2\pi R\mathbb{Z}\})$.

Theorem

This CFT verifies Segal's axioms. Roughly speaking, it defines a projective functor from the cobordism category of Riemann surfaces to the category of Hilbert spaces.

The full theory is developed in [Guillarmou–Kupiainen–Rhodes '23, `arXiv:2310.18226`] for the ordinary CFT and [Xiao–X. '25, to appear] for the boundary CFT.



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Structure Constants

The 3-point correlation function on the sphere $\widehat{\mathbb{C}}$ is

$$\langle V_{\alpha_1}(0)V_{\alpha_2}(1)V_{\alpha_3}(\infty)\rangle_{\widehat{\mathbb{C}},g_0} = C_{g_0}\, 2\pi R\, \frac{(-\mu)^p}{p!}\, C_\beta^{\mathrm{ImDOZZ}}(\alpha_1,\alpha_2,\alpha_3)$$

where C_{g_0} is an explicit constant depending on the metric $g_0 = \max(|z|,1)^{-4}\,|dz|^2$ and

$$C_\beta^{\mathrm{ImDOZZ}}(\alpha_1,\alpha_2,\alpha_3) =$$
$$\left(\frac{\beta}{2}\right)^{Q(2Q-\bar{\alpha})}\left(\frac{\pi\Gamma(1-\frac{\beta^2}{4})}{\Gamma(\frac{\beta^2}{4})}\right)^p \frac{\Upsilon_{\frac{\beta}{2}}(Q+\frac{\beta}{2}-\frac{\bar{\alpha}}{2})\prod_{j=1}^3\Upsilon_{\frac{\beta}{2}}(\alpha_j+\frac{\beta}{2}-\frac{\bar{\alpha}}{2})}{\Upsilon_{\frac{\beta}{2}}(\frac{\beta}{2})\prod_{j=1}^3\Upsilon_{\frac{\beta}{2}}(\frac{2}{\beta}+\alpha_j)}$$

is the **imaginary DOZZ constant**. Here we used the shorthands $\bar{\alpha} = \sum\alpha_j$, $p = \frac{2Q-\bar{\alpha}}{\beta} \in \mathbb{N}_{\geq 0}$.

This constant also appears as certain statistics for percolations and Conformal Loop Ensembles (see, e.g., [Ang–Cai–Sun–Wu '24, `arXiv:2107.01788`]).

For the Boundary CFT, the structure constants on the disk $\mathbb{D}$ involve integrals of Dotsenko–Fateev–Selberg type:

$$\int_{\mathbb{D}^p}\int_{\partial\mathbb{D}^q}\prod_{j=1}^{p}|w_j|^{\alpha\beta}|1-w_j|^{\eta\beta}(1-|w_j|^2)^{\beta^2/2}\prod_{j<j'}|w_j-w_{j'}|^{\beta^2}|1-w_j\bar{w}_{j'}|^{\beta^2}$$
$$\prod_{k=1}^{q}|1-y_k|^{\eta\beta/2}\prod_{k<k'}|y_k-y_{k'}|^{\beta^2/2}\prod_{j,k}|w_j-y_k|^{\beta^2}\,\mathrm{d}y\,\mathrm{d}w$$

where $p,q\in\mathbb{N}$, $2p+q=2Q-2\alpha-\eta$. This is solved only in special cases.

A Special Case

The boundary 1-point structure constant with bulk $\mu = 0$ is

$$\int_{\partial\mathbb{D}^q} \prod_{k=1}^{q} |1 - y_k|^{\eta\beta/2} \prod_{k<k'} |y_k - y_{k'}|^{\beta^2/2} \mathrm{d}y = M(\eta\beta/4, \eta\beta/4, \beta^2/4)$$

where

$$M(\eta\beta/4, \eta\beta/4, \beta^2/4) = \prod_{j=0}^{q-1} \frac{\Gamma(1 + \eta\beta/2 + j\beta^2/4)\Gamma(1 + (j+1)\beta^2/4)}{\Gamma(1 + \eta\beta/4 + j\beta^2/4)^2\Gamma(1 + \beta^2/4)}$$

When $\eta = 0$, this becomes the Fyodorov–Bouchaud formula

$$\frac{\Gamma(1 + q\beta^2/4)}{\Gamma(1 + \beta^2/4)^p}.$$

Algebraic Structure

- This is a non-unitary CFT.
- Its Hamiltonian $\mathbf{H}$ has compact resolvent with discrete spectrum

$$\operatorname{Spec}\mathbf{H} = \left\{ \sqrt{\frac{n^2}{R^2} + R^2 k^2 + 2j} : n, k, j \in \mathbb{N}_{\geq 0} \right\} \subset \mathbb{R}$$

  which coincides with the free theory.
- However, $\mathbf{H}$ is not diagonalizable and it is possible to construct explicit Jordan blocks.

Outlook

- Find all structure constants of Boundary CILT
- Clarify the Virasoro structure
- Solve the conformal bootstrap
- Connections with Generalized Minimal Models, Critical Loop Models, Schramm–Loewner Evolution, Conformal Loop Ensembles, ...